

Bounds on Chromatic Polynomials

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Abstract

In this paper, we find a new phenomenon on chromatic polynomials of graphs. Let $\chi_G(t) = a_0 t^n - a_1 t^{n-1} + \cdots (-1)^r a_r t^{n-r}$ be the chromatic polynomial of a simple graph G . For any $q, k \in \mathbb{Z}$ with $0 \leq k \leq \min\{r, q + r + 1\}$, we show that the partial binomial sum $\sum_{i=0}^k \binom{q}{i} a_{k-i}$ of a_i is bounded above by $\binom{m+q}{k}$ and below by $\binom{r+q}{k}$, i.e.,

$$\binom{r+q}{k} \leq \sum_{i=0}^k \binom{q}{i} a_{k-i} \leq \binom{m+q}{k}.$$

Keywords: Chromatic polynomials, characteristic polynomials, hyperplane arrangements, binomial sum.

1 Introduction

First appeared in [1], the chromatic polynomial $\chi_G(t)$ counts the number of proper colorings of a simple graph G with t colors. Write

$$\chi_G(t) = a_0 t^n - a_1 t^{n-1} + \cdots (-1)^r a_r t^{n-r},$$

where $n = |VG|$ is the number of vertices of G and $r = n -$ the number of components of G , called the rank of G . The chromatic polynomial is one of the most central topics in graph theory. In 1932, Whitney [7] gave a combinatorial interpretation to the number a_i , which counts the number of i -subsets of edges of G containing no broken circuits, known as the *broken circuit theorem*. He [8] further showed that the coefficient sequence is sign-alternating, i.e., $a_i > 0$. In 1968, Read [5] asked which polynomial is the chromatic polynomial of some graph and conjectured that the sequence a_0, a_1, \dots, a_n is unimodal. In 1970, G.H.J. Meredith [3] gave an upper bound for each coefficient, i.e., $|a_i| \leq \binom{m}{i}$, where $m = |EG|$ is the number of edges of G . Most recently, Huh [2] showed that the coefficient sequence a_0, a_1, \dots, a_n is logconcave, which resolved the conjecture on the unimodality by Read. Here we shall present a new phenomenon on the coefficient sequence a_0, a_1, \dots, a_n , which gives another necessary condition on coefficients of chromatic polynomials of graphs.

The main result can be easily stated as follows. If $q, k \in \mathbb{Z}$ with $0 \leq k \leq \min\{r, q + r + 1\}$, then

$$\binom{r+q}{k} \leq \sum_{i=0}^k \binom{q}{i} a_{k-i} \leq \binom{m+q}{k}.$$

As a direct consequence, Whitney's sign-alternating theorem and Meredith's upper bound theorem can be obtained by specializing $q = 0$, i.e.,

$$\binom{r}{k} \leq a_k \leq \binom{m}{k}, \quad \text{for } 0 \leq k \leq r.$$

Moreover, we can see that the partial sum of the coefficient sequence of the chromatic polynomial is still sign-alternating and bounded by $\binom{m-1}{k}$ after taking $q = -1$, i.e.,

$$\binom{r-1}{k} \leq (-1)^k \sum_{i=0}^k (-1)^i a_i \leq \binom{m-1}{k}, \quad \text{for } 0 \leq k \leq r-1.$$

Since the characteristic polynomial of hyperplane arrangements can be viewed as a generalization of the chromatic polynomial of graphs, we shall prove all above results for the characteristic polynomial of hyperplane arrangements.

2 Bounds on Coefficients of Characteristic Polynomials

An n -dimensional arrangement \mathcal{A} of hyperplanes is a collection of finite hyperplanes in an n -dimensional vector space V , called the hyperplane arrangement for short. The intersection semi-lattice $L(\mathcal{A})$ of \mathcal{A} is defined to be the collection of all nonempty intersections of hyperplanes in \mathcal{A} , with the partial order given by the inverse of set inclusion, that is,

$$L(\mathcal{A}) = \{\cap \mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A}\},$$

where $\cap \mathcal{B}$ is an abbreviate of $\cap_{H \in \mathcal{B}} H$. The minimal element of $L(\mathcal{A})$ is assumed to be $\hat{0} = \cap_{H \in \emptyset} H := V$. Denote by r the rank function of $L(\mathcal{A})$. The rank of the maximal element in the semilattice $L(\mathcal{A})$ is called the *rank* of \mathcal{A} , denoted $r(\mathcal{A})$, which is also the dimension of the vector space spanned by normals of all hyperplanes in \mathcal{A} . We call the hyperplane arrangement \mathcal{A} *central* if the intersection $\cap \mathcal{A}$ of all hyperplanes in \mathcal{A} is nonempty. Define the *characteristic polynomial* $\chi(\mathcal{A}, t) \in \mathbb{C}[t]$ of \mathcal{A} to be

$$\chi(\mathcal{A}; t) := \sum_{X \in L(\mathcal{A})} \mu(\hat{0}, X) t^{\dim(X)}.$$

where μ is the Möbius function of $L(\mathcal{A})$. It is well known that the characteristic polynomial satisfies the *deletion-contraction* recurrence, i.e., for any $H \in \mathcal{A}$, we have

$$\chi(\mathcal{A}; t) = \chi(\mathcal{A} \setminus H; t) - \chi(\mathcal{A}/H; t),$$

where $\mathcal{A} \setminus H$ is the subarrangement of \mathcal{A} in V obtained by removing H from \mathcal{A} , and \mathcal{A}/H is a hyperplane arrangement in H whose members are restrictions of all elements of $\mathcal{A} \setminus H$ on H , i.e.,

$$\mathcal{A} \setminus H := \mathcal{A} \setminus \{H\}, \quad \mathcal{A}/H := \{H' \cap H \mid H' \in \mathcal{A} \setminus H\}.$$

One of the most important hyperplane arrangements is the *graphic arrangement*. Suppose $G = (VG, EG)$ is a simple graph with the vertex set $VG = [n]$ and the edge set $EG \subseteq [n] \times [n]$. Then G determines an n -dimensional arrangement \mathcal{A}_G of $|EG|$ hyperplanes whose members are given by

$$H_{ij} : x_i = x_j, \quad \text{for } (i, j) \in EG.$$

From Theorem 2.7 in [6], we have that

$$\chi(\mathcal{A}_G; t) = \chi_G(t).$$

An arrangement \mathcal{A} of hyperplanes is called the *boolean arrangement* if $r(\mathcal{A} \setminus H) < r(\mathcal{A})$ for all $H \in \mathcal{A}$, or equivalently $r(\mathcal{A}) = |\mathcal{A}|$. It is obvious that the graphic arrangement \mathcal{A}_G is boolean if and only if the graph G is a forest. A hyperplane arrangement \mathcal{A} of rank r is called *in general position* if it satisfies the following two conditions,

- if $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| \leq r$, then $\cap \mathcal{B} \neq \emptyset$ and $r(\cap \mathcal{B}) = |\mathcal{B}|$;
- if $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| > r$, then $\cap \mathcal{B} = \emptyset$.

The intersection semi-lattice of a hyperplane arrangement in general position is a truncated boolean lattice. From Proposition 2.3, we can see that the boolean arrangement is a hyperplane arrangement in general position. From [6, Proposition 2.4], the characteristic polynomial of an n -dimensional arrangement \mathcal{A} of m hyperplanes in general position with $r(\mathcal{A}) = r$ is

$$\chi(\mathcal{A}; t) = t^n - mt^{n-1} + \binom{m}{2} t^{n-2} - \binom{m}{3} t^{n-3} - \dots + (-1)^r \binom{m}{r} t^{n-r}. \quad (1)$$

Indeed, we shall prove in Proposition 2.2 that the converse of the above statement is still true, which requires the broken-circuit Theorem 2.1. Let $\mathcal{A} = \{H_1, \dots, H_m\}$ be an n -dimensional arrangement

of hyperplanes. We call the subset \mathcal{B} of \mathcal{A} **dependent** if $\cap \mathcal{B} \neq \emptyset$ and $r(\cap \mathcal{B}) < |\mathcal{B}|$. Suppose that \mathcal{A} be totally ordered by any given order \prec . A **circuit** of \mathcal{A} is a minimal dependent subset of \mathcal{A} , namely, a dependent subset which is not dependent after removing any element from it. A **broken circuit** is a subset of \mathcal{A} obtained by removing the maximal element from a circuit of \mathcal{A} . A subset \mathcal{B} of \mathcal{A} is called χ -**independent** if $\cap \mathcal{B} \neq \emptyset$ and \mathcal{B} contains no broken circuits.

Theorem 2.1 (Broken-circuit Theorem [4]). *Let \mathcal{A} be an n -dimensional arrangement of rank r with the characteristic polynomial*

$$\chi(\mathcal{A}; t) = a_0 t^n - a_1 t^{n-1} + \cdots + (-1)^r a_r t^{n-r}.$$

Then a_k is the number of χ -independent k -subsets of \mathcal{A} for $0 \leq k \leq r$.

Proposition 2.2. *Let \mathcal{A} be an n -dimensional arrangement of m hyperplanes with $r(\mathcal{A}) = r$. Then \mathcal{A} is in general position if and only if its characteristic polynomial is the formula (1).*

Proof. Note that no subset of \mathcal{A} is a circuit since \mathcal{A} is in general position. it follows that every i -subset of \mathcal{A} is χ -independent for $i \leq r$. Then the broken-circuit Theorem 2.1 implies $a_i = \binom{m}{i}$ for $i \leq r$, which proves the direction \Rightarrow . Conversely, suppose that the characteristic polynomial of \mathcal{A} is of form (1), i.e., $a_i = \binom{|\mathcal{A}|}{i}$ for all $i \leq r$. Then all i -subsets of \mathcal{A} are χ -independent for $i \leq r$. If $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| \leq r$, then $\cap \mathcal{B} \neq \emptyset$ and \mathcal{B} contains no broken circuits. Since that the rank of \mathcal{A} is r , the broken circuit of \mathcal{A} has cardinality at most r . It implies that \mathcal{A} contains no broken circuits. So \mathcal{A} does not contain circuits. It follows that $|\mathcal{B}| = r(\cap \mathcal{B})$ if $|\mathcal{B}| \leq r$ and $\cap \mathcal{B} = \emptyset$ otherwise, which completes the proof. \square

Proposition 2.3. *An n -dimensional arrangement \mathcal{A} of m hyperplanes is boolean if and only if its intersection semi-lattice $L(\mathcal{A})$ is the boolean lattice $(2^{[m]}, \subseteq)$. Hence, the boolean arrangement \mathcal{A} is central and its characteristic polynomial is*

$$\chi(\mathcal{A}; t) = t^{n-m}(t-1)^m = \sum_{i=0}^m (-1)^i \binom{m}{i} t^{n-i}.$$

Proof. Let X be a maximal element in the intersection semi-lattice $L(\mathcal{A})$. Suppose that there is a hyperplane $H \in \mathcal{A}$ which does not pass through X , i.e., $X \not\subseteq H$. Then we have $r(\mathcal{A} \setminus H) = r(\mathcal{A}) = r(X)$, which contradicts that \mathcal{A} is boolean. It follows that $X \subseteq H$ for all $H \in \mathcal{A}$. Hence, \mathcal{A} is central. Without loss of generality, assume that \mathcal{A} is linear, i.e., all hyperplanes of \mathcal{A} contains origin. Then the hypothesis $r(\mathcal{A} \setminus H) < r(\mathcal{A})$ implies that the normal vectors of all hyperplanes $H \in \mathcal{A}$ are linear independent. It implies that the intersection semi-lattice is the boolean lattice. So the characteristic polynomial can be easily obtained from the möbius function of the boolean lattice. The converse statement is obvious. \square

Theorem 2.4. *Let \mathcal{A} be an n -dimensional arrangement of m hyperplanes and rank r with the characteristic polynomial*

$$\chi(\mathcal{A}; t) = a_0 t^n - a_1 t^{n-1} + a_2 t^{n-2} + \cdots + (-1)^r a_r t^{n-r}.$$

If $q, k \in \mathbb{Z}$ with $0 \leq k \leq \min\{r, q+r+1\}$, then

$$\binom{r+q}{k} \leq \sum_{i=0}^k \binom{q}{i} a_{k-i} \leq \binom{m+q}{k}. \quad (2)$$

In particular, the coefficients of the chromatic polynomial $\chi_G(t)$ of a graph G satisfy the inequality (2).

Proof. We first prove the following polynomial identity,

$$\sum_{i=0}^k \binom{x}{i} \binom{y}{k-i} = \binom{x+y}{k}, \quad \text{for } k \in \mathbb{Z}_{\geq 0}, \quad (3)$$

where $\binom{x}{i} = \frac{1}{i!}x(x-1)\cdots(x-i+1)$. For any fixed $x \in \mathbb{C}$, we claim that (3) holds for all nonnegative integers y . This claim can be proved by using induction on $y \in \mathbb{Z}_{\geq 0}$. The induction basis $y = 0$ is trivial. For any $y \in \mathbb{Z}_{\geq 0}$, the claim is followed by

$$\sum_{i=0}^k \binom{x}{i} \binom{y+1}{k-i} = \sum_{i=0}^k \binom{x}{i} \binom{y}{k-i} + \sum_{i=0}^{k-1} \binom{x}{i} \binom{y}{k-1-i} = \binom{x+y}{k} + \binom{x+y}{k-1} = \binom{x+y+1}{k},$$

where the second equality is obtained from the induction hypothesis. Note that (3) forms a polynomial equation in y when x is fixed, which implies (3) holds for all $x, y \in \mathbb{C}$. Otherwise, the equation (3) has finite many solutions of $y \in \mathbb{C}$ in terms of the fixed x , which contradicts to the claim that all nonnegative integers y are solutions of (3) for each fixed x . We next use induction on $m = |\mathcal{A}|$ to prove this theorem. If the arrangement \mathcal{A} is boolean, it is trivial by Proposition 2.3 and the identity (3). Assume that there is an element $H \in \mathcal{A}$, such that $r = r(\mathcal{A} \setminus H) = r(\mathcal{A})$. Then we can write

$$\begin{aligned} \chi(\mathcal{A} \setminus H; t) &= b_0 t^n - b_1 t^{n-1} + b_2 t^{n-2} + \cdots + (-1)^r b_r t^{n-r}, \\ \chi(\mathcal{A}/H; t) &= c_0 t^{n-1} - c_1 t^{n-2} + c_2 t^{n-3} + \cdots + (-1)^{r-1} c_{r-1} t^{n-r}. \end{aligned}$$

From the deletion-contraction recurrence, we have $a_i = b_i + c_{i-1}$ for $0 \leq i \leq r$, where $c_{-1} = 0$. Notice that $|\mathcal{A} \setminus H| = m - 1$ and $r - 1 \leq |\mathcal{A}/H| \leq m - 1$. By the induction hypothesis, we have

$$\binom{r-1+q}{k} \leq \binom{r+q}{k} \leq \sum_{i=0}^k \binom{q}{i} b_{k-i} \leq \binom{m-1+q}{k}; \quad \text{if } q+r+1 \geq k,$$

and

$$\binom{r-1+q}{k-1} \leq \sum_{i=0}^{k-1} \binom{q}{i} c_{k-1-i} \leq \binom{|\mathcal{A}/H|+q}{k-1} \leq \binom{m-1+q}{k-1}; \quad \text{if } q+r \geq k-1.$$

The inequality (2) can be obtained by summing the above formulae together. Since $\chi_G(t) = \chi(\mathcal{A}_G; t)$ for the graph G , the second part of the theorem is an easy consequence of the first one. \square

Remarks:

1. If the condition $q+r+1 \geq k$ is relaxed to $q+r+2 \geq k$, we won't have the inequality $\binom{r-1+q}{k} \leq \binom{r+q}{k}$ in the proof of Theorem 2.4. For instance, this inequality is not true for $k=2$ and $r+q=0$.
2. It is easily seen that the case $q \geq 0$ of the formula (2) can be inductively obtained from the case $q=0$. Hence the essential part of (2) is the case $q \leq 0$.
3. Indeed, the inequality (2) holds for the chromatic polynomial $\chi(M; t)$ of any matroid M , since that $\chi(M; t)$ also satisfies the deletion-contraction recurrence which is the only fact we need to use in the proof of Theorem 2.4.

Corollary 2.5. *With the same assumptions as Theorem 2.4, for $0 \leq k \leq r$, we have*

- (i) $\binom{r}{k} \leq a_k \leq \binom{m}{k}$;
- (ii) $\binom{r-1}{k} \leq (-1)^k \sum_{i=0}^k (-1)^i a_i \leq \binom{m-1}{k}$;
- (iii) $0 \leq \sum_{i=0}^k \binom{k-r-1}{i} a_{k-i} \leq \binom{m+k-r-1}{k}$.

Proof. (i), (ii), and (iii) are direct consequences of Theorem 2.4 by taking $q=0$, $q=-1$, and $q=k-r-1$ respectively. \square

Obviously, Corollary 2.5 (i) implies both Whitney's sign-alternating theorem [8] and Meredith's upper bound theorem [3]. Moreover, by Proposition 2.3, the left equality of (i) holds if and only if the arrangement \mathcal{A} is boolean, and by Proposition 2.2, the right equality of (i) holds if and only if the arrangement \mathcal{A} is in general position. Corollary 2.5 (ii) implies that the sequence of the partial sum $\sum_{i=0}^k (-1)^i a_i$ ($0 \leq k < r$) of the coefficients of the characteristic polynomial is also sign-alternating. Corollary 2.5 (iii) gives the substantial inequalities of (2), i.e., the inequality in Corollary 2.5 (iii) implies the inequality (2). This statement can be proved combinatorially by using induction on q which we won't present in this paper.

Theorem 2.6. Let $\chi_G(t) = t^n - a_1 t^{n-1} + \cdots + (-1)^r a_r t^{n-r}$ be the chromatic polynomial of a graph G with n vertices, m edges, and rank r . Then the following three statements are equivalent,

- (i) $a_k = \binom{m}{k}$ for all k with $1 \leq k \leq r$;
- (ii) $a_k = \binom{r}{k}$ for all k with $1 \leq k \leq r$;
- (iii) G is a forest, i.e., $m = r$.

Proof. Since \mathcal{A}_G is central, it follows that for any $\mathcal{B} \subseteq \mathcal{A}_G$, we have $\cap \mathcal{B} \neq \emptyset$. From Proposition 2.2, $a_k = \binom{m}{k}$ if and only if the graphic arrangement \mathcal{A}_G is in general position. It follows that $|\mathcal{A}_G| = |EG| = m = r$ which proves (i) \Leftrightarrow (iii). (iii) \Rightarrow (ii) can be easily obtained by applying Corollary 2.5 (i). And (ii) \Rightarrow (iii) is trivial according to the fact $a_1 = |EG| = m$. \square

3 Further Discussions and Open Problems

Let \mathcal{A} be an n -dimensional arrangement of m hyperplanes and rank r . If all hyperplanes of \mathcal{A} are restricted onto the r -dimensional subspace spanned by their normal vectors, we shall obtain an r -dimensional arrangement of m hyperplanes and rank r , whose characteristic polynomial has the same coefficient sequence as $\chi(\mathcal{A}; t)$. Hence we can assume that the characteristic polynomial of \mathcal{A} is

$$\chi(\mathcal{A}; t) = a_0 t^r - a_1 t^{r-1} + \cdots + (-1)^r a_r. \quad (4)$$

Recall the broken-circuit Theorem 2.1 that a_k counts the number of χ -independent k -subsets of \mathcal{A} . It is then obvious that $a_k \leq \binom{m}{k}$. On the other hand, $a_r \neq 0$ implies that there exists at least one χ -independent r -subset \mathcal{B} of \mathcal{A} . Note the fact from the definition that any subset of a χ -independent set is still χ -independent. Then all subsets of \mathcal{B} are χ -independent, which implies $a_k \geq \binom{r}{k}$ by the broken-circuit Theorem 2.1. In this sense, the inequality $\binom{r}{k} \leq a_k \leq \binom{m}{k}$ of Corollary 2.5 can be interpreted from the broken-circuit theorem. We then naturally propose the following question.

- Can the inequality (2) be interpreted by the broken-circuit theorem?

Define an operation D of the characteristic polynomial $\chi(\mathcal{A}; t)$ to be

$$D\chi(\mathcal{A}; t) = \frac{\chi(\mathcal{A}; t) - \chi(\mathcal{A}; 1)}{t - 1}.$$

Denote by $D^0 = \text{id}$ the identity map and $D^i \chi(\mathcal{A}; t) = \underbrace{D \circ \cdots \circ D}_i \chi(\mathcal{A}; t)$ for $i \in \mathbb{N}$. Note that

$$\begin{aligned} \chi(\mathcal{A}; t) - \chi(\mathcal{A}; 1) &= a_0(t^r - 1) - a_1(t^{r-1} - 1) + \cdots + (-1)^{r-1} a_{r-1}(t - 1) \\ &= (t - 1) \left(\sum_{k=0}^{r-1} t^{r-1-k} \sum_{i=0}^k (-1)^i a_i \right) \end{aligned}$$

Then we have

$$D\chi(\mathcal{A}; t) = \sum_{k=0}^{r-1} \left(\sum_{i=0}^k (-1)^i a_i \right) t^{r-1-k}$$

In general, for $q \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} D^q \chi(\mathcal{A}; t) &= \sum_{k=0}^{r-q} (-1)^k \left(\sum_{i=0}^k \binom{-q}{i} a_{k-i} \right) t^{r-q-k} \\ &= t^{r-q} - (m - q) t^{r-q-1} + \cdots + (-1)^{r-q} \sum_{i=0}^{r-q} \binom{-q}{i} a_{r-q-i}. \end{aligned}$$

If $D^q \chi(\mathcal{A}; t)$ can be geometrically realized as the characteristic polynomial of a hyperplane arrangement of $m - q$ hyperplanes and rank $r - q$, from discussions at the beginning of this section, the

inequality (2) can be easily interpreted by the broken-circuit theorem, which answers the previous question. So our question is reduced to a geometric realization of $D^q\chi(\mathcal{A};t)$ as the characteristic polynomial of an arrangement of hyperplanes for all $q \in \mathbb{Z}_{\geq 0}$. Indeed, this question can be further reduced as follows

- Is there an arrangement of hyperplanes whose characteristic polynomial is $D\chi(\mathcal{A};t)$?

Suppose we can find a geometric realization to $D\chi(\mathcal{A};t)$ for any \mathcal{A} , i.e., $D\chi(\mathcal{A};t) = \chi(\mathcal{A}_1;t)$ for some hyperplane arrangement \mathcal{A}_1 . Similarly, we shall have a hyperplane arrangement \mathcal{A}_2 such that $D\chi(\mathcal{A}_\infty;t) = \chi(\mathcal{A}_2;t)$. Continuing this process, the geometric realization of $D^q\chi(\mathcal{A};t)$ can be obtained reductively for all $q \in \mathbb{Z}_{\geq 0}$. Hence, the positive answer of the above question combining with the broken-circuit theorem will provide an intuitive interpretation to the inequality (2).

When \mathcal{A} is linear, i.e., all hyperplanes in \mathcal{A} pass through the origin, we can construct an affine hyperplane arrangement $d\mathcal{A}$ such that $\chi(d\mathcal{A};t) = D\chi(\mathcal{A};t)$. Suppose \mathcal{A} is a linear arrangement of $m+1$ hyperplanes in \mathbb{R}^n , i.e., all hyperplanes in \mathcal{A} pass through the origin, given $K_0 \in \mathcal{A}$ with the defining equation $\sum_{i=1}^n \alpha_i x_i = 0$, the **deconing** $d\mathcal{A}$ of \mathcal{A} is an arrangement of m hyperplanes in the affine space $K_1 : \sum_{i=1}^n \alpha_i x_i = 1$, which is defined by

$$d\mathcal{A} = \{H \cap K_1 \mid H \in \mathcal{A}, H \neq K_0\}.$$

It is well known that

$$\chi(d\mathcal{A};t) = D\chi(\mathcal{A};t),$$

since $\chi(\mathcal{A},1) = 0$ when \mathcal{A} is linear. Namely, the deconing construction can geometrically realize $D\chi(\mathcal{A};t)$ as the characteristic polynomial $\chi(d\mathcal{A};t)$ of the hyperplane arrangement $d\mathcal{A}$ for the linear hyperplane arrangement \mathcal{A} . However, this construction can not directly apply to affine cases. It be interesting to find such a geometric construction for an affine hyperplane arrangement \mathcal{A} such that $\chi(d\mathcal{A};t) = D\chi(\mathcal{A};t)$.

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